TRINITY COLLEGE

Year 12 Mathematics Specialist 3/4
Test 2 2022

* Scientific Calculator ONLY

Functions and Sketching Graphs

STUDENT'S NAME Solutions PRESSER

DATE: Thursday 24 March **TIME:** 50 minutes **MARKS**: 52

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (5 marks)

The functions *f* and *g* are defined by $f(x) = 7x - 1$ and $g(x) = \dfrac{4}{x-2}$.

(a) Solve for *x* if $f \circ g(x) = x$. [3]

$f \circ g(x) = f\left(\frac{4}{x-2}\right)$

$= \dfrac{28}{x-2} - 1$

Now $\dfrac{28}{x-2} - 1 = x$

$\Rightarrow 28 - (x-2) = x^2 - 2x$

$\Rightarrow x^2 - x - 30 = 0$

$\Rightarrow (x+5)(x-6) = 0$

$\Rightarrow x = -5$ or 6

(b) Determine the largest value of *a* such that $g(a) = f^{-1}(a)$. [2]

let $y = 7x - 1$

So $\dfrac{y+1}{7} = x$

$\Rightarrow f^{-1}(x) = \dfrac{x+1}{7}$

Now $\dfrac{4}{a-2} = \dfrac{a+1}{7}$

$\Rightarrow 28 = (a+1)(a-2)$

$\Rightarrow 0 = a^2 - a - 30$

$\Rightarrow a = 6$ (from (a))

2. (6 marks)

(a) Determine the **two** discontinuities associated with the graph of the function $f(x)=\dfrac{x^2-4x+3}{x^2-x-6}$. [3]

$= \dfrac{(x-3)(x-1)}{(x-3)(x+2)}$

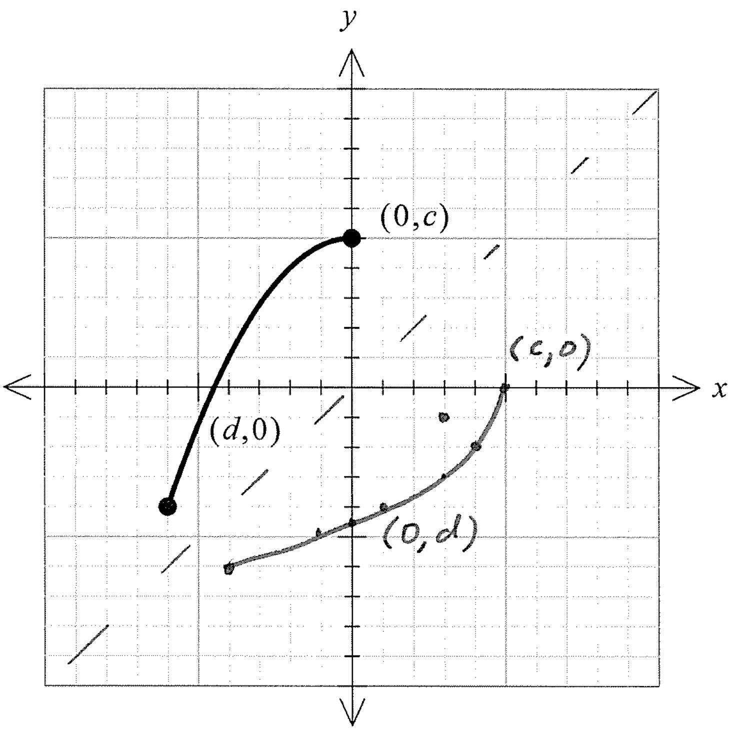
$= \dfrac{x-1}{x+2}$

So at

$x=-2$, vertical asymptote

$x=3$, hole

(b) The graph below shows a sketch of the curve with equation $y=g(x),\ x\le 0$. The curve has intercepts at $(0,c)$ and $(d,0)$.



(i) Explain why $g(x)$ has an inverse function $g^{-1}(x)$. [1]

$g(x)$ is a one-to-one function because it passes the horizontal line test.

(ii) Sketch the graph of $g^{-1}(x)$ on the axes above clearly indicating the coordinates of the x and y intercepts. [2]

3. (7 marks)

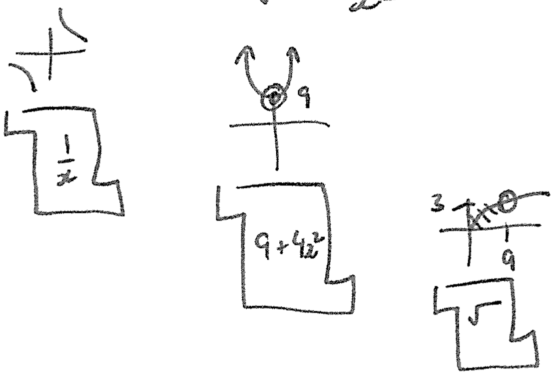
(a) For $f(x)=\sqrt{9+4x^2}$ and $g(x)=\dfrac{1}{x}$, determine the domain and range of $f\circ g(x)$. [3]

$f\circ g(x) = f\left(\frac{1}{x}\right)$

$= \sqrt{9+\frac{4}{x^2}}$

$D_{f\circ g}\ \{x : x \in \mathbb{R},\ x \neq 0\}$

$R_{f\circ g}\ \{y : y \in \mathbb{R},\ y > 3\}$



OR

$9 + \frac{4}{x^2} > 0$

always true except

$x = 0$

(b) A rational function P is defined by $P(x)=\dfrac{ax+b}{x+c}$. The graph of $P(x)$ has the following features:

- An x-intercept at $x=-\dfrac{1}{2}$
- A horizontal asymptote of $y=2$
- A vertical asymptote of $x=5$

Determine the values of a, b and c. [4]

Vertical asymptote $x=5$ $\Rightarrow$ $c=-5$

Horizontal asymptote $y=2$ $\Rightarrow$ as $x\to\pm\infty$ $P\to a$

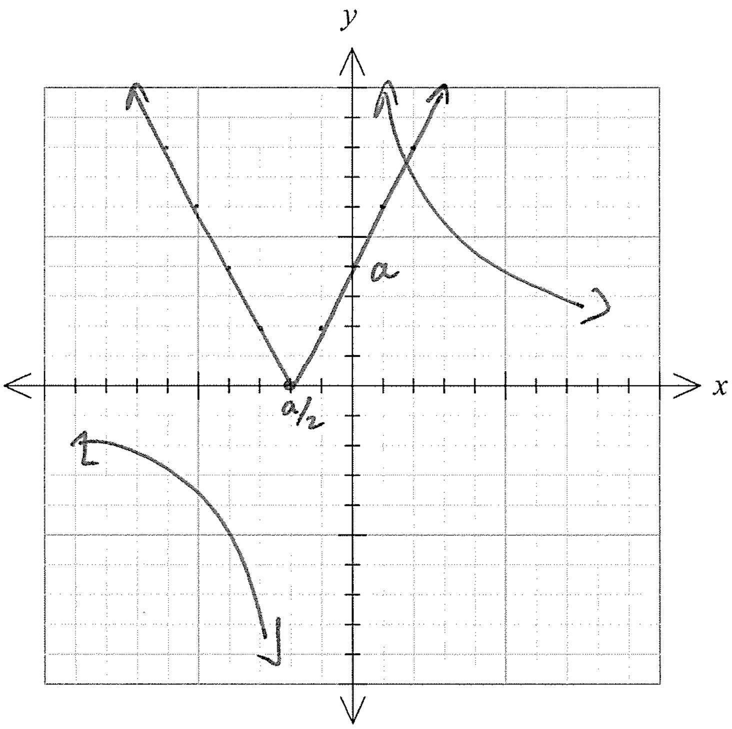
$\Rightarrow$ $a=2$

Pt $\left(-\frac{1}{2}, 0\right)$ on curve $\Rightarrow$ $0 = \dfrac{2\left(-\frac{1}{2}\right)+b}{-\frac{1}{2}-5}$

$\Rightarrow$ $b = 1$

4. (7 marks)

(a) Sketch the graph of $y=|2x+a|$, $a>0$ on the axes below showing the coordinates of the points where the graph meets the coordinate axes. [2]



$y = |2x+a|$

$= 2|x+\frac{a}{2}|$

(b) On the same axes, sketch the graph of $y=\frac{1}{x}$. [1]

(c) Explain how your graphs show that there is only one solution to the equation $x|2x+a|-1=0$. [1]

$\Rightarrow \quad |2x+a| = \frac{1}{x}$

Graphs only intersect once.

(d) Determine the value of x for which $x|2x+\cancel{4}|-1=0$. [3]

$\Rightarrow \quad |2x+\cancel{4}| = \frac{1}{x}$

$\Rightarrow \quad 2x+1 = \frac{1}{x} \quad (\text{as } x > -\frac{1}{2})$

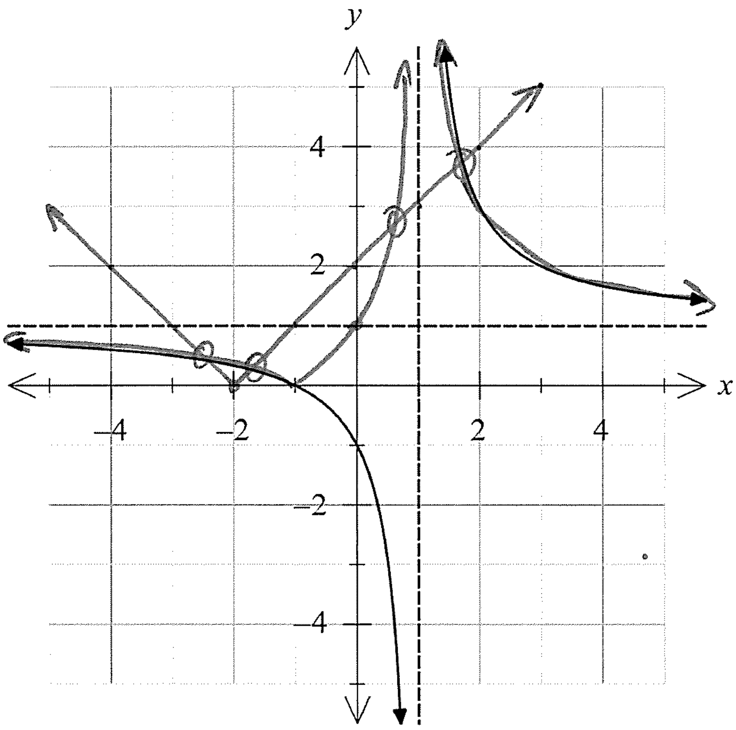
$\Rightarrow \quad 2x^2+x-1=0$

$\Rightarrow \quad (2x-1)(x+1)=0$

$\Rightarrow \quad x = \frac{1}{2} \quad \text{as } x > -\frac{1}{2}$

5. (5 marks)

The graph of $y = f(x) = \dfrac{x+1}{x-1}$ is drawn on the axes below.



(a) Sketch on the axes above the graph of $y = |f(x)|$. [2]

(b) Sketch on the same axes the graph of $y = |x+2|$. [2]

(c) Hence, state the number of solutions to $\left|\dfrac{x+1}{x-1}\right| = |x+2|$. You are not required to calculate the solutions. [1]

There are 4 points of intersection.

$\therefore$ 4 solns.

6. (14 marks)

The curve C has equation $f(x)=\dfrac{(x-1)^2}{x+1}$. $= \dfrac{x^2-2x+1}{x+1} = x-3+\dfrac{4}{x+1}$

(a) Determine the equations of the asymptotes of C. [3]

$$\begin{array}{r} x-3 \\ x+1\,\overline{)\,x^2-2x+1} \\ \underline{x^2+x} \\ -3x+1 \\ \underline{-3x-3} \\ 4 \end{array}$$

$\therefore\ f(x) = x-3+\dfrac{4}{x+1}$

So vertical asymptote $x=-1$

oblique asymptote $y=x-3$

(b) Determine the intercepts of C. [2]

y-int $(0,1)$ from subbing $x=0 \Rightarrow \frac{(-1)^2}{1}$

x-int $(1,0)$ from numerator $(x-1)^2=0$

(c) Show that C has two stationary points. Determine their coordinates and nature. [4]

$f(x) = x-3+\dfrac{4}{x+1}$

$f'(x) = 1-\dfrac{4}{(x+1)^2}$

tp when $f'(x)=0$

$\Rightarrow\ 0 = 1-\dfrac{4}{(x+1)^2}$

$\Rightarrow\ (x+1)^2 = 4$

$\Rightarrow\ x+1 = \pm 2$

$\Rightarrow\ x = -3$ or 1

Coordinates $(-3,-8)$ and $(1,0)$

Max Min

$f'(x) = 1-4(x+1)^{-2}$

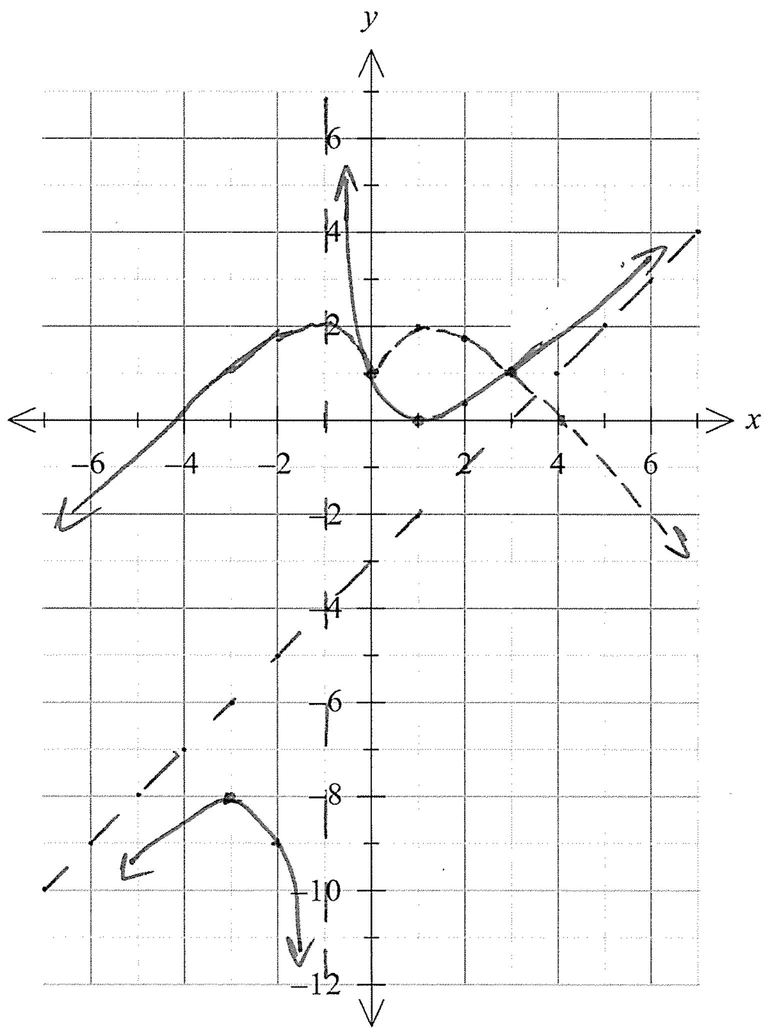
$f''(x) = 8(x+1)^{-3}$

$= \dfrac{8}{(x+1)^3}$

$f''(-3) = -ve$

$f''(1) = +ve$

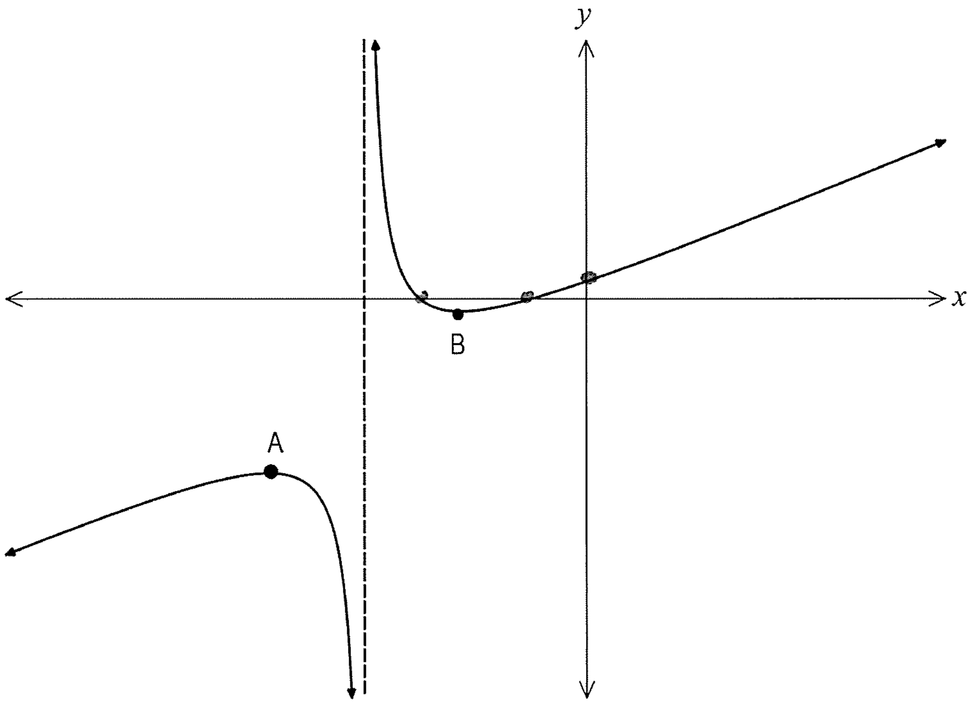
(d) Draw a sketch of C on the axes below. [2]



(e) On the same axes above draw a sketch of the curve $y = -f(|x|) + 2$. [3]

7. (7 marks)

The graph below is a sketch of $y = \dfrac{(x+1)(x+m)}{x+4}$.



(a) Determine the equation of the vertical asymptote. [1]

$x = -4$

(b) Determine the coordinates of the three points where the graph cuts the axes. [3]

y-int $\left(0, \frac{m}{4}\right)$

x-int $(-1, 0)$ and $(-m, 0)$

(c) Given that the points A and B are the only stationary points on the curve, determine any restrictions on the value of m. Justify your answer. [3]

[Hint: The function can be written as $y = x + (m-3) + \dfrac{12-3m}{x+4}$.]

$$y = x + (m-3) + \frac{12-3m}{x+4}$$

$$y' = 1 - \frac{(12-3m)}{(x+4)^2}$$

Only two stationary pts. So

$$0 = 1 - \frac{12-3m}{(x+4)^2}$$

$$\Rightarrow \quad 12 - 3m = (x+4)^2$$

$$\Rightarrow \quad x+4 = \pm\sqrt{12-3m}$$

For the eqn to have two real solns,

$$\Rightarrow \quad 12 - 3m > 0$$

$$\Rightarrow \quad 12 > 3m$$

$$\Rightarrow \quad m < 4$$

For the eqn to have a +ve y-int, $m > 0$